

# Binary matroids and local complementation

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## Abstract

We introduce a binary matroid  $M(IAS(G))$  associated with a looped simple graph  $G$ .  $M(IAS(G))$  classifies  $G$  up to local equivalence, and determines the delta-matroid and isotropic system associated with  $G$ . Moreover, a parametrized form of its Tutte polynomial yields the interlace polynomials of  $G$ .

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Mathematics Subject Classification. 05C50

## 1 Introduction

A graph  $G = (V(G), E(G))$  consists of a finite vertex-set  $V(G)$  and a finite edge-set  $E(G)$ . Each edge is incident on one or two vertices; an edge incident on only one vertex is a *loop*. The two vertices incident on a non-loop edge are *neighbors*, and the *open neighborhood* of a vertex  $v$  is  $N(v) = \{\text{neighbors of } v\}$ . A graph in which different edges can be distinguished by their vertex-incidences is a *looped simple graph*, and a *simple graph* is a looped simple graph with no loop.

In this paper we are concerned with properties of looped simple graphs motivated by two sets of ideas. The first set of ideas is the theory of the principal pivot transform (PPT) over  $GF(2)$ . PPT over arbitrary fields was introduced more than 50 years ago by Tucker [34]; see also the survey of Tsatsomeros [33]. According to Geelen [22], PPT transformations applied to the mod-2 adjacency matrices of looped simple graphs are generated by two kinds of *elementary* PPT operations, *non-simple local complementations* with respect to looped vertices and *edge pivots* with respect to edges connecting unlooped vertices. The second set of ideas is the theory of 4-regular graphs and their Euler circuits, initiated more than 40 years ago by Kotzig [24]. Kotzig proved that all the Euler circuits of a 4-regular graph are obtained from any one using  $\kappa$ -transformations. If a 4-regular graph is directed in such a way that every vertex has indegree 2 and

outdegree 2, then Kotzig [24], Pevzner [26] and Ukkonen [35] showed that all of the graph's directed Euler circuits are obtained from any one through certain combinations of  $\kappa$ -transformations called *transpositions* by Arratia, Bollobás and Sorkin [2, 3, 4]. Bouchet [7] and Rosenstiehl and Read [27] introduced a simple graph associated with any Euler circuit of a connected 4-regular graph, the *alternance* graph or *interlacement* graph; an equivalent *link relation matrix* was defined by Cohn and Lempel [18] in the context of the theory of permutations. These authors showed that the effects of  $\kappa$ -transformations and transpositions on interlacement graphs are given by *simple* local complementations and edge pivots, respectively.

In the late 1980s, Bouchet introduced two new kinds of combinatorial structures associated with these two theories. On the one hand are the *delta-matroids* [8], some of which are associated with looped simple graphs. The fundamental operation of delta-matroid theory is a way of changing one delta-matroid into another, called *twisting*. Two looped simple graphs are related through PPT operations if and only if their associated delta-matroids are related through twisting. On the other hand are the *isotropic systems* [9, 11], all of which are associated with *fundamental graphs*. Two isotropic systems are *strongly isomorphic* if and only if they share fundamental graphs. Moreover, two simple graphs are related through simple local complementations if and only if they are fundamental graphs of strongly isomorphic isotropic systems. Properties of isotropic systems were featured in the proof of Bouchet's famous "forbidden minors" characterization of circle graphs [13].

The purpose of this paper is to introduce a binary matroid  $M(IAS(G))$  constructed in a natural way from the adjacency matrix of a looped simple graph  $G$ ; we call it the *isotropic matroid of  $G$* , in honor of Bouchet's isotropic systems. This matroid directly determines both the delta-matroid and the isotropic system associated with  $G$ . Moreover, it is not difficult to characterize the effects of edge pivots and both kinds of local complementations on isotropic matroids, so we have a single matroid invariant that serves to classify  $G$  under all the operations mentioned above.

The paper is set up as follows. First, we recall some basic facts about binary matroids in Section 2.  $M(IAS(G))$  is defined in Section 3, where we also discuss a certain kind of matroid isomorphism between isotropic matroids. The heart of the paper is Section 4, where we show that  $M(IAS(G))$  determines the equivalence classes of  $G$  under various kinds of complementations and pivots. In Section 5 we show that the delta-matroid and isotropic system associated with  $G$  are determined by  $M(IAS(G))$ . In the last section we show that  $M(IAS(G))$  has another interesting property: appropriately parametrized Tutte polynomials of this matroid yield the *interlace polynomials* introduced by Arratia, Bollobás and Sorkin [2, 3, 4], and also the modified versions subsequently defined by Aigner and van der Holst [1], Courcelle [19] and the author [31].

The ideas in this paper came to mind after the resemblance between the matrices appearing in Aigner and van der Holst's discussion of interlace polynomials [1] and our nonsymmetric approach to interlacement in 4-regular graphs [32] was pointed out to us by R. Brijder. We are grateful to him for years of

informative correspondence regarding delta-matroids, isotropic systems, PPT and related combinatorial notions.

## 2 Standard representations of binary matroids

We do not review general results and terminology of graph theory and matroid theory here; instead we refer the reader to standard texts in the field, [23, 25, 36, 37] for instance. All the matroids we consider in this paper are *binary*:

**Definition 1** *Let  $S$  be a finite set. A binary matroid  $M$  on  $S$  is represented by a matrix with entries in  $GF(2)$ , whose columns are indexed by the elements of  $S$ . A subset of  $S$  is dependent in  $M$  if and only if the corresponding columns of the matrix are linearly dependent.*

The binary matroid represented by a matrix is not changed if one row is added to another, or the rows are permuted, or a row of zeroes is adjoined or removed. Also, permuting the columns of a matrix will yield a new matrix that represents an isomorphic binary matroid. Familiar results of elementary linear algebra tell us that consequently, every binary matroid has a representation of the following type:

**Definition 2** *Let  $I$  be an  $r \times r$  identity matrix. A standard representation of a rank- $r$  binary matroid  $M$  is a matrix of the form  $(I \mid A)$  that represents  $M$ .*

If  $A$  is a matrix with entries in  $GF(2)$  then  $M(IA)$  denotes the matroid with standard representation  $(I \mid A)$ .

Recall that if  $B$  is a basis of a matroid  $M$ , and  $x$  is an element of  $M$  not included in  $B$ , then the *fundamental circuit* of  $x$  with respect to  $B$  is

$$C(x, B) = \{x\} \cup \{b \in B \mid B \Delta \{b, x\} \text{ is a basis of } M\},$$

where  $\Delta$  denotes the symmetric difference.  $C(x, B)$  is the unique circuit contained in  $B \cup \{x\}$ .

A peculiar property of binary matroids is that the fundamental circuits with respect to any one basis contain enough information to determine a binary matroid. The same is not true for general matroids; for instance a matroid on  $\{1, 2, 3, 4\}$  with basis  $\{1, 2\}$  and fundamental circuits  $\{1, 2, 3\}$  and  $\{1, 2, 4\}$  might be either  $U_{2,4}$  or the circuit matroid of a triangle with one doubled edge. ( $U_{2,4}$  is not binary, of course.) Notice that in essence, a standard representation  $(I \mid A)$  is this kind of description: the matroid elements corresponding to the columns of  $I$  constitute a basis  $B$ , and for each element  $x \notin B$ , the fundamental circuit  $C(x, B)$  includes  $x$  together with the elements of  $B$  corresponding to nonzero entries of the  $x$  column of  $A$ .

The only part of this section that does not appear in the textbooks mentioned above is the following simple theorem, which tells us how the various standard representations of a binary matroid are related to each other.

**Theorem 3** Let  $A_1$  and  $A_2$  be  $r \times (n-r)$  matrices with entries in  $GF(2)$ . Then  $M(IA_1) \cong M(IA_2)$  if and only if  $(I \mid A_2)$  can be obtained from  $(I \mid A_1)$  using the following three types of operations on matrices of the form  $(I \mid A)$ :

- (a) Permute the columns of  $A$ .
- (b) Permute the columns of  $I$  and the rows of  $(I \mid A)$ , using the same permutation.
- (c) Suppose the  $jk$  entry of  $A$  is  $a_{jk} = 1$ . Then toggle (reverse)  $a_{bc}$  whenever  $b \neq j$ ,  $c \neq k$ ,  $a_{jc} = 1$  and  $a_{bk} = 1$ .

**Proof.** As noted above, a standard presentation of a rank- $r$  binary matroid  $M$  on an  $n$ -element set  $S$  is obtained as follows. First choose a basis  $B$ , and index its elements as  $s_1, \dots, s_r$ . Then index the remaining elements of  $S$  as  $s_{r+1}, \dots, s_m$ . Finally, let  $A$  be the  $r \times (n-r)$  matrix whose  $jk$  entry is 1 if and only if  $s_j$  is an element of the fundamental circuit  $C(s_{r+k}, B)$ .

Operations of types (a) and (b) correspond to re-indexings of  $B$  and  $S - B$ , respectively.

Suppose now that  $a_{jk} = 1$ , and let  $A'$  be the matrix obtained from  $A$  by an operation of type (c). Another way to describe  $A'$  is this:  $(I \mid A')$  is obtained from  $(I \mid A)$  by first interchanging the  $j$ th and  $(r+k)$ th columns, and then adding the  $j$ th row of the resulting matrix to every other row in which the original  $(r+k)$ th column has a nonzero entry. That is, the matrix  $(I \mid A')$  is simply the standard presentation corresponding to the basis  $B\Delta\{s_j, s_{r+k}\}$ , with the elements other than  $s_j$  and  $s_{r+k}$  indexed as they were before.

The theorem follows, because basis exchanges  $B \mapsto B\Delta\{b, x\}$  eventually construct every basis of  $M$  from any one. ■

We refer to an operation of type (c) as a *basis exchange* involving the  $j$ th column of  $I$  and the  $k$ th column of  $A$ . (It would also be natural to call it a *pivot*, but this term already has other meanings.)

### 3 $M(IAS(G))$ and compatible isomorphisms

If  $G$  is a looped simple graph then  $A(G)$  denotes the adjacency matrix of  $G$ , i.e., the  $|V(G)| \times |V(G)|$  matrix with entries in  $GF(2)$  given by: a diagonal entry is 1 if and only if the corresponding vertex is looped, and an off-diagonal entry is 1 if and only if the corresponding vertices are adjacent.  $AS(G)$  denotes the matrix  $(A(G) \mid I + A(G))$ . ( $S$  is for “sum.”)

As in Section 2,  $M(IAS(G))$  is the binary matroid represented by the  $|V(G)| \times (3|V(G)|)$  matrix

$$IAS(G) = (I \mid A(G) \mid I + A(G)).$$

We denote the ground set of this matroid  $W(G)$ . If  $v \in V(G)$  then there are three columns of  $IAS(G)$  corresponding to  $v$ : one in  $I$ , one in  $A(G)$ , and one in  $I + A(G)$ . For notational convenience, and to indicate the connection with our work on interlace polynomials [31], we use  $v_\phi$  to denote the column of  $I$  corresponding to  $v$ ,  $v_\chi$  to denote the column of  $A(G)$  corresponding to  $v$ , and

$v_\psi$  to denote the column of  $I + A(G)$  corresponding to  $v$ . We refer to the resulting partition of  $W(G)$  into 3-element subsets as *canonical*.

It is convenient to adopt notation to describe matroid isomorphisms that are compatible with these canonical partitions. Let  $S_3$  denote the group of permutations of the three symbols  $\phi$ ,  $\chi$  and  $\psi$ . We use standard notation in  $S_3$ : for instance 1 is the identity,  $(\phi\chi)$  is a transposition, and  $(\phi\chi)(\chi\psi) = (\psi\phi\chi)$  is a 3-cycle.

Suppose  $G_1$  and  $G_2$  are looped simple graphs, and there is an isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  that is compatible with the canonical partitions. Then the isomorphism consists of two parts. First, there is an induced bijection  $V(G_1) \rightarrow V(G_2)$ ; in general we will denote this bijection  $\beta$  too, though up to isomorphism we may always presume that  $V(G_1) = V(G_2)$  and the induced bijection is the identity map. Second, there is a function  $f_\beta : V(G_1) \rightarrow S_3$  such that  $\beta(v_\iota) = \beta(v)_{f_\beta(\iota)} \forall v \in V(G_1) \forall \iota \in \{\phi, \chi, \psi\}$ . In this situation we say that  $\beta$  is a *compatible isomorphism determined by  $f_\beta$* .

Here are two obvious properties of compatible isomorphisms.

**Lemma 4** *If  $\beta_1 : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  and  $\beta_2 : M(IAS(G_2)) \rightarrow M(IAS(G_3))$  are compatible isomorphisms, then  $\beta_2 \circ \beta_1 : M(IAS(G_1)) \rightarrow M(IAS(G_3))$  is also a compatible isomorphism, and it is determined by the map  $f : V(G_1) \rightarrow S_3$  given by  $f(v) = f_{\beta_2}(v) \cdot f_{\beta_1}(v)$ .*

**Lemma 5** *If  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  is a compatible isomorphism then so is  $\beta^{-1} : M(IAS(G_2)) \rightarrow M(IAS(G_1))$ , and  $\beta^{-1}$  is determined by the map  $f_{\beta^{-1}} : V(G_2) \rightarrow S_3$  given by  $f_{\beta^{-1}}(v) = f_\beta(v)^{-1}$ .*

The next property is not quite so obvious.

**Lemma 6** *Suppose  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  is a compatible isomorphism, determined by the map  $f : V(G_1) \rightarrow S_3$  with  $f(v) = 1 \forall v \in V(G_1)$ . Then  $G_1$  and  $G_2$  are isomorphic.*

**Proof.** Up to isomorphism, we may as well presume that  $V(G_1) = V(G_2)$  and the bijection  $V(G_1) \rightarrow V(G_2)$  induced by  $\beta$  is the identity map. Then  $M(IAS(G_1))$  and  $M(IAS(G_2))$  are isomorphic matroids on the ground set  $W(G_1) = W(G_2)$ . As  $f(v) \equiv 1$ , the isomorphism  $\beta$  is the identity map of this ground set.

The identity map preserves the basis  $B = \{v_\phi \mid v \in V(G_1)\}$ . The identity map is a matroid isomorphism, so it must also preserve fundamental circuits with respect to  $B$ . Recall the discussion of Section 2: the column of  $IAS(G_i)$  corresponding to  $x \notin B$  is determined by the fundamental circuit of  $x$  with respect to  $B$  in  $M(IAS(G_i))$ . It follows that the matrices  $AS(G_1)$  and  $AS(G_2)$  are identical. ■

The following consequence will be useful.

**Corollary 7** *Suppose  $\beta_1 : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  and  $\beta_2 : M(IAS(G_1)) \rightarrow M(IAS(G_3))$  are compatible isomorphisms, and the associated functions  $f_{\beta_1}, f_{\beta_2} : V(G_1) \rightarrow S_3$  are the same. Then  $G_2$  and  $G_3$  are isomorphic.*

**Proof.** Apply the lemmas to the composition of  $\beta_2$  and  $\beta_1^{-1}$ . ■

## 4 Complements and pivots

In this section we prove that the matroid  $M(IAS(G))$  classifies  $G$  under several different kinds of operations.

### 4.1 Loop complementation

Suppose  $G_1$  is a looped simple graph,  $v \in V(G_1)$ , and  $G_2$  is the graph obtained from  $G_1$  by complementing (reversing) the loop status of  $v$ . Clearly then  $IAS(G_2)$  is the matrix obtained from  $IAS(G_1)$  by interchanging the  $v_\chi$  and  $v_\psi$  columns. This interchange is an example of an operation of type (a), so Theorem 3 tells us that there is a compatible isomorphism  $M(IAS(G_1)) \rightarrow M(IAS(G_2))$  determined by the map  $f : V(G_1) \rightarrow S_3$  given by

$$f(w) = \begin{cases} \text{the transposition } (\chi\psi), & \text{if } w = v \\ 1, & \text{if } w \neq v \end{cases}.$$

The converse also holds:

**Theorem 8** *Let  $G_1$  and  $G_2$  be looped simple graphs, and suppose  $v \in V(G_1)$ . Then these two conditions are equivalent:*

1. *Up to isomorphism,  $G_2$  is the graph obtained from  $G_1$  by complementing the loop status of  $v$ .*
2. *There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  such that  $f_\beta(v) = (\chi\psi)$  and  $f_\beta(w) = 1 \ \forall w \neq v$ .*

**Proof.** We have already discussed the implication  $1 \Rightarrow 2$ . The converse follows from Corollary 7. ■

### 4.2 Local complementation

Two different versions of *local complementation* appear in the literature. *Simple* local complementation was introduced by Bouchet [7] and Rosenstiehl and Read [27], as part of the theory of interlacement in 4-regular graphs. This operation does not involve the creation of loops, so it is the version seen most often in graph theory, where the theory of simple graphs predominates. *Non-simple* local complementation is part of the theory of the principal pivot transform (PPT) over  $GF(2)$ . The general theory of PPT was introduced by Tucker [34]; see also the survey of Tsatsomeros [33]. The special significance of non-simple local complementation in PPT over  $GF(2)$  was discussed by Geelen [22]. Later (and independently) non-simple local complementation was introduced by Arratia, Bollobás and Sorkin as part of the theory of the two-variable interlace polynomial [4]. We should emphasize that simple local complementations are usually applied only to simple graphs in the first set of references, and non-simple local complementations are usually applied only with respect to looped vertices in the second set of references. Our definitions are not so restrictive.

**Definition 9** *If  $G$  is a looped simple graph and  $v \in V(G)$  then the simple local complement of  $G$  with respect to  $v$  is the graph  $G_s^v$  obtained from  $G$  by complementing all adjacencies between distinct elements of the open neighborhood  $N(v)$ . The non-simple local complement of  $G$  with respect to  $v$  is the graph  $G_{ns}^v$  obtained from  $G_s^v$  by complementing the loop status of each element of  $N(v)$ .*

Observe that replacing  $A(G)$  by  $A(G_{ns}^v)$  has precisely the same effect on the matrix  $IAS(G)$  as a type (c) operation from Theorem 3. As discussed in Section 2, this operation is equivalent to a basis exchange involving  $v_\phi$  and either  $v_\chi$  (if  $v$  is looped) or  $v_\psi$  (if  $v$  is unlooped). We deduce the following.

**Theorem 10** *Let  $G_1$  and  $G_2$  be looped simple graphs, and suppose  $v \in V(G_1)$ . Then these two conditions are equivalent:*

1.  $G_2$  is isomorphic to  $(G_1)_{ns}^v$ .
2. There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  such that  $f_\beta(w) = 1 \ \forall w \neq v$  and

$$f_\beta(v) = \begin{cases} (\phi_\chi), & \text{if } v \text{ is looped in } G_1 \\ (\phi_\psi) & \text{if } v \text{ is not looped in } G_1 \end{cases}.$$

**Proof.** As already noted,  $1 \Rightarrow 2$  because  $IAS((G_1)_{ns}^v)$  is the same as the matrix associated with the standard presentation of  $M(IAS(G_1))$  obtained from  $IAS(G_1)$  by a basis exchange involving  $v_\phi$  and either  $v_\chi$  or  $v_\psi$ . The converse follows from Corollary 7. ■

### 4.3 Pivots

Here is a well-known definition. The reader who is encountering it for the first time should take a moment to verify that the two indicated triple local complements are indeed the same.

**Definition 11** *If  $v$  and  $w$  are neighbors in  $G$  then the edge pivot  $G^{vw}$  is the triple simple local complement:*

$$G^{vw} = ((G_s^v)^w)_s^v = ((G_s^w)^v)_s^w.$$

Note that we do not restrict edge pivots to edges with unlooped vertices.

**Corollary 12** *Suppose  $v \neq w$  are neighbors in  $G$ , and let  $f : V(G) \rightarrow S_3$  be the function with  $f(x) = 1$  for  $w \neq x \neq v$ ,*

$$f(v) = \begin{cases} (\phi_\chi), & \text{if } v \text{ is unlooped} \\ (\phi_\psi), & \text{if } v \text{ is looped} \end{cases} \quad \text{and} \quad f(w) = \begin{cases} (\phi_\chi), & \text{if } w \text{ is unlooped} \\ (\phi_\psi), & \text{if } w \text{ is looped} \end{cases}.$$

*Then  $f$  determines a compatible isomorphism  $M(IAS(G)) \rightarrow M(IAS(G^{vw}))$ .*

**Proof.** The reader can easily check that the definition

$$G^{vw} = ((G_s^v)^w)_s^v$$

is equivalent to saying this:  $G^{vw}$  is obtained from  $((G_{ns}^v)^w)_{ns}^v$  by complementing the loop status of  $v$ . (The reason is that  $v$  is the only vertex whose loop status is complemented an odd number of times in  $((G_{ns}^v)^w)_{ns}^v$ .) It follows that there is a compatible isomorphism  $M(IAS(G)) \rightarrow M(IAS(G^{vw}))$  obtained by composing four compatible isomorphisms, three from Theorem 12 and one from Theorem 8. According to Lemma 4, this compatible isomorphism is determined by the function  $f : V(G) \rightarrow S_3$  such that  $f(x) = 1$  for  $w \neq x \neq v$ ,  $f(w) = (\phi\psi)$  if  $w$  is unlooped in  $G_{ns}^v$ ,  $f(w) = (\phi\chi)$  if  $w$  is looped in  $G_{ns}^v$ , and

$$f(v) = \begin{cases} (\chi\psi) \cdot (\phi\chi) \cdot 1 \cdot (\phi\psi), & \text{if } v \text{ is unlooped} \\ (\chi\psi) \cdot (\phi\psi) \cdot 1 \cdot (\phi\chi), & \text{if } v \text{ is looped} \end{cases}.$$

■

As in Theorems 8 and 12, Corollary 7 implies that the converse of Corollary 12 is also valid. That is, if the given function  $f$  determines a compatible isomorphism  $M(IAS(G_1)) \rightarrow M(IAS(G_2))$  then  $G_2 \cong G_1^{vw}$ .

Notice that the function  $f$  of Corollary 12 is the combination of two separate functions, one  $\equiv 1$  except at  $v$  and the other  $\equiv 1$  except at  $w$ . According to Theorem 3, these two separate functions do not come from two separate compatible isomorphisms, though. This fact is reflected in the proof, where the compatible isomorphism  $M(IAS(G)) \rightarrow M(IAS(G^{vw}))$  is described as a composition of *four* simpler compatible isomorphisms, not two.

There is a different way to describe the compatible isomorphism of Corollary 12, using only two basis exchanges. According to Theorem 3, if  $v$  and  $w$  are neighbors then there is a basis exchange involving  $v_\phi$  and either  $w_\chi$  or  $w_\psi$ , as both these columns have 1s in the  $v$  row. The matrix resulting from part (c) of Theorem 3 is not of the form  $(I \mid A \mid I + A)$  for a symmetric matrix  $A$ , so there is no natural way to interpret such a basis exchange as a graph operation. However, the reader can easily check that if this basis exchange is followed by one involving  $w_\phi$  and either  $v_\chi$  or  $v_\psi$ , then the result *is* of the form  $(I \mid A \mid I + A)$  for a symmetric matrix  $A$ . Moreover,  $A$  closely resembles the adjacency matrix of  $G^{vw}$ ; the positions of  $v$  and  $w$  have been interchanged, though, and depending on the  $\chi, \psi$  choices the loop statuses of  $v$  and  $w$  may also have changed. The fact that the compatible isomorphism  $M(IAS(G)) \rightarrow M(IAS(G^{vw}))$  may be described in this different way, involving a transposition of adjacency information regarding  $v$  and  $w$ , is a reflection of the fact that there is a different way to define the pivot. See Section 3 of [3], where Arratia, Bollobás and Sorkin give this different definition, and show that it is related to Definition 11 by applying a “label swap” exchanging the names of  $v$  and  $w$ .



#### 4.4 Classification theorems

The simplest classification theorem resulting from the above discussion is this immediate consequence of Theorem 8.

**Theorem 13** *Let  $G_1$  and  $G_2$  be looped simple graphs. Then these two conditions are equivalent:*

1. *Up to isomorphism,  $G_2$  can be obtained from  $G_1$  by complementing the loop status of some vertices.*
2. *There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  such that  $f_\beta(v) \in \{1, (\chi\psi)\} \forall v \in V(G_1)$ .*

Other classification theorems have similar statements, but take a little more work to prove.

**Theorem 14** *Let  $G_1$  and  $G_2$  be looped simple graphs. Then these two conditions are equivalent:*

1. *Up to isomorphism,  $G_2$  can be obtained from  $G_1$  using edge pivots.*
2. *There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  such that  $f_\beta(v) \in \{1, (\phi\psi)\}$  for every looped  $v \in V(G_1)$  and  $f_\beta(v) \in \{1, (\phi\chi)\}$  for every unlooped  $v \in V(G_1)$ .*

**Proof.** If  $G'$  can be obtained from  $G$  using edge pivots, simply apply Corollary 12 repeatedly.

For the converse, suppose condition 2 holds, and there are  $k$  vertices with  $f_\beta(v) \neq 1$ . If  $k = 0$  then Corollary 7 implies that  $G_1 \cong G_2$ .

If  $k > 0$  then let  $v_0 \in V(G)$  have  $f_\beta(v_0) \neq 1$ , and let  $B = \{v_\phi \mid v \in V(G_1)\}$ . Then  $B$  is a basis of  $M(IAS(G_1))$ , so  $\beta(B)$  is a basis of  $M(IAS(G_2))$ . Consequently,

$$\{\beta(v_{0\phi})\} \cup \{v_\phi \mid f_\beta(v_\phi) = 1\}$$

cannot be dependent in  $M(IAS(G_2))$ , because it is a subset of  $\beta(B)$ . It follows that the column of  $IAS(G_2)$  corresponding to  $\beta(v_{0\phi})$  must have at least one nonzero entry in a row that corresponds to a vertex  $v \neq v_0$  with  $f_\beta(v) \neq 1$ . Then  $v$  is a neighbor of  $v_0$ , and Lemma 12 implies that there is a compatible isomorphism  $\beta' : M(IAS(G_2)) \rightarrow M(IAS((G_2)^{vv_0}))$  determined by the function  $f_{\beta'} : V(G_2) \rightarrow S_3$  with  $f_{\beta'}(w) = 1 \forall w \notin \{v, v_0\}$ ,  $f_{\beta'}(v_0) = f_\beta(v_0)$  and  $f_{\beta'}(v) = f_\beta(v)$ . The composition  $\beta' \circ \beta$  is a compatible isomorphism  $M(IAS(G_1)) \rightarrow M(IAS((G_2)^{vv_0}))$ , which satisfies condition 2; as there are only  $k - 2$  vertices outside  $(f_{\beta' \circ \beta})^{-1}(1)$ , induction assures us that up to isomorphism,  $(G_2)^{vv_0}$  may be obtained from  $G_1$  using edge pivots. ■

When restricted to simple graphs, Theorem 14 yields the following.

**Corollary 15** *Let  $G_1$  and  $G_2$  be simple graphs. Then these two conditions are equivalent:*

1. Up to isomorphism,  $G_2$  can be obtained from  $G_1$  using edge pivots.
2. There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  such that  $f_\beta(v) \in \{1, (\phi\chi)\} \forall v \in V(G_1)$ .

Another consequence of Theorem 14 is this:

**Corollary 16** *Let  $G_1$  and  $G_2$  be looped simple graphs. Then these three conditions are equivalent:*

1. Up to isomorphism,  $G_2$  can be obtained from  $G_1$  using two kinds of operations: non-simple local complementations with respect to looped vertices, and edge pivots with respect to non-looped vertices.
2. Up to isomorphism,  $G_2$  can be obtained from  $G_1$  using PPT operations.
3. There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  such that  $f_\beta(v) \in \{1, (\phi\chi)\} \forall v \in V(G_1)$ .

**Proof.** The equivalence  $1 \Leftrightarrow 2$  is due to Geelen [22], and the implication  $1 \Rightarrow 3$  follows from Theorems 10 and 14.

Suppose condition 3 holds and there are  $k$  vertices with  $f_\beta(v) \neq 1$ . If  $k = 0$  then Corollary 7 implies that  $G_1$  and  $G_2$  are isomorphic. If  $G_1$  has a looped vertex  $v_0$  with  $f_\beta(v_0) \neq 1$  then there is a compatible isomorphism  $\beta' : M(IAS((G_1)_{ns}^{v_0})) \rightarrow M(IAS(G_2))$  that satisfies condition 3, and for which only  $k - 1$  vertices have  $f_{\beta'}(v) \neq 1$ . Induction then tells us that condition 1 holds. If there is no such  $v_0$ , then Corollary 15 applies. ■

Our final classification theorem is this.

**Theorem 17** *Let  $G_1$  and  $G_2$  be looped simple graphs. Then these two conditions are equivalent:*

1. Up to isomorphism,  $G_2$  can be obtained from  $G_1$  using local complementations and loop complementations.
2. There is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$ .

**Proof.** The implication  $1 \Rightarrow 2$  follows from Theorems 8 and 10.

For the converse, suppose condition 2 holds and there are  $k$  vertices with  $f_\beta(v) \neq 1$ . Up to isomorphism, we may presume that  $V(G_1) = V(G_2)$  and the bijection induced by  $\beta$  is the identity map.

If  $k = 0$  then Corollary 7 tells us that  $G_1 = G_2$ .

The argument proceeds by induction on  $k \geq 1$ . If there is any vertex with  $f_\beta(v_0) = (\chi\psi)$  then the graph  $G'_2$  obtained from  $G_2$  by complementing the loop status of  $v_0$  has the property that there is a compatible isomorphism  $\beta' : M(IAS(G_1)) \rightarrow M(IAS(G'_2))$  such that  $f_{\beta'}(v) = f_\beta(v) \forall v \neq v_0$ , and  $f_{\beta'}(v_0) = 1$ . The inductive hypothesis tells us that up to isomorphism,  $G'_2$  can be obtained from  $G_1$  using local complementations and loop complementations. Of course we can then obtain  $G_2$  from  $G'_2$  by loop complementation, so condition 1 holds.

If there is a looped vertex  $v_0$  with  $f_\beta(v_0) = (\phi\chi)$  or an unlooped vertex  $v_0$  with  $f_\beta(v_0) = (\phi\psi)$ , then a similar argument applies, with  $G'_2 = (G_2)_{ns}^{v_0}$ .

If there is a looped vertex  $v_0$  with  $f_\beta(v_0) = (\phi\psi)$  or an unlooped vertex  $v_0$  with  $f_\beta(v_0) = (\phi\chi)$ , then the same argument used in the proof of Theorem 14 tells us that there is a vertex  $v$  that neighbors  $v_0$  in  $G_2$  and has  $f_\beta(v) \neq 1$ . Then the graph  $G'_2 = (G_2)^{v_0 v}$  has the property that there is a compatible isomorphism  $\beta' : M(IAS(G_1)) \rightarrow M(IAS(G'_2))$  such that  $f_{\beta'}(w) = f_\beta(w) \forall w \notin \{v_0, v\}$ , and  $f_{\beta'}(v_0) = 1$ . The inductive hypothesis tells us that up to isomorphism,  $G'_2$  can be obtained from  $G_1$  using local complementations and loop complementations; of course we can then obtain  $G_2$  from  $G'_2$  using local complementations.

Finally, if there is a vertex  $v_0$  with  $f_\beta(v_0)$  a 3-cycle then the graph  $G'_2$  obtained from  $G_2$  by complementing the loop status of  $v_0$  has the property that there is a compatible isomorphism  $\beta' : M(IAS(G_1)) \rightarrow M(IAS(G'_2))$  such that  $f_{\beta'}(v) = f_\beta(v) \forall v \neq v_0$ , and  $f_{\beta'}(v_0) = (\chi\psi) \cdot f_\beta(v_0)$  is a transposition. Consequently one of the preceding arguments applies to  $G'_2$ . ■

**Corollary 18** *Let  $G_1$  and  $G_2$  be simple graphs. Then  $G_1$  can be obtained from  $G_2$  using simple local complementations if and only if there is an isomorphism*

$$M(IAS(G_1)) \cong M(IAS(G_2))$$

*that is compatible with the canonical partitions.*

#### 4.5 $M(IA(G))$

If  $G$  is a looped simple graph then we call the binary matroid  $M(IA(G))$  the *restricted isotropic matroid* of  $G$ ; it is represented by the  $|V(G)| \times (2|V(G)|)$  matrix

$$IA(G) = (I \mid A(G)).$$

Suppose  $G_1$  and  $G_2$  are looped simple graphs, and there is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  with  $f_\beta(v) \in \{1, (\phi\chi)\} \forall v \in V(G_1)$ . Then  $\beta(v_\psi) = \beta(v)_\psi \forall v \in V(G_1)$ , so  $\beta$  restricts to an isomorphism between the submatroids  $M(IA(G_1))$  and  $M(IA(G_2))$ . Moreover, this restriction of  $\beta$  is compatible with the natural partitions of these matroids into pairs, and the restriction determines  $\beta$ .

Corollary 15 implies that a simple graph is classified up to pivot equivalence by compatible isomorphisms of  $M(IA(G))$ . Similarly, Corollary 16 implies that a looped simple graph is classified up to PPT equivalence by compatible isomorphisms of  $M(IA(G))$ .

Compatible isomorphisms with  $f_\beta(v)(\phi) \equiv \phi$  are less interesting, as Theorem 8 tells us that they can be realized using loop complementation.

A compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  with  $f_\beta(v)(\chi) = \chi \forall v$  can be analyzed by first applying loop complementation to all vertices in both  $G_1$  and  $G_2$ , and then analyzing the corresponding compatible isomorphism  $\beta' : M(IAS(G'_1)) \rightarrow M(IAS(G'_2))$ , which has  $f_{\beta'}(v)(\psi) = \psi \forall v \in V(G'_1)$ .

## 5 Delta-matroids and isotropic systems

The results of this paper show that the theory of binary matroids contains “conceptual imbeddings” of the theories of graphic delta-matroids and isotropic systems, two interesting and useful theories studied by Bouchet in the 1980s and 1990s. Bouchet later introduced a third theory, involving *multimatroids*, to unify these two. Using terminology of [14], we can summarize our “conceptual imbeddings” by saying two things. First, if  $G$  is a looped simple graph then  $M(IAS(G))$  is a binary matroid that shelters the 3-matroid associated with an isotropic system with fundamental graph  $G$ , and the submatroid  $M(IA(G))$  shelters the 2-matroid associated with a delta-matroid with fundamental graph  $G$ . (“Sheltering” is a way of containing; that’s why we use the term “imbedding.”) Second, matroidal properties of  $M(IAS(G))$  provide new explanations of the properties of graphic delta-matroids and isotropic systems; that’s what makes the imbeddings “conceptual.” For instance, compatible isomorphisms of isotropic matroids provide a new explanation of the significance of isotropic systems, using the fact that certain kinds of basis exchanges correspond to local complementations. They also provide a new way to conceptualize the work of Brijder and Hoogeboom [16] on the connection between  $S_3$  and certain operations on delta-matroids.

**Definition 19** [10] *If  $G$  is a looped simple graph, then the delta-matroid associated to  $G$  is*

$$D(G) = \{S \subseteq V(G) \mid \text{the submatrix } A(G)[S] \text{ is nonsingular over } GF(2)\}.$$

Here  $A(G)[S]$  denotes the principal submatrix of  $A(G)$  obtained by removing all rows and columns corresponding to vertices  $v \notin S$ . Observe that

$$D(G) = \{S \subseteq V(G) \mid \{s_\chi \mid s \in S\} \cup \{v_\phi \mid v \notin S\} \text{ is a basis of } M(IAS(G))\},$$

so the matroid  $M(IAS(G))$  determines  $D(G)$ . (The index  $\psi$  does not appear in this description of  $D(G)$ , so the submatroid  $M(IA(G))$  actually contains enough information to determine  $D(G)$ .) Moreover, if  $G_1$  and  $G_2$  are looped simple graphs and there is a compatible isomorphism  $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$  with  $f_\beta(v)(\psi) = \psi \forall v \in V(G_1)$ , then the set  $X = \{v \in V(G_1) \mid f_\beta(v) \neq 1\}$  has the property that

$$D(G_2) = \{S \Delta X \mid S \in D(G_1)\}.$$

Consequently, the significance of symmetric difference (also called “twisting”) for the theory of graphic delta-matroids follows from the results of Section 4, regarding compatible isomorphisms of  $M(IAS(G))$  and  $M(IA(G))$ .

It takes a little more work to see how  $M(IAS(G))$  determines an isotropic system.

**Definition 20** *If  $G$  is a looped simple graph then the sub-transversals of  $W(G)$  are the elements of  $\mathcal{S}(W(G)) = \{S \subseteq W(G) \mid |S \cap \{v_\phi, v_\chi, v_\psi\}| \leq 1 \forall v \in V(G)\}$ .*

The following is easily proven.

**Proposition 21**  $\mathcal{S}(W(G))$  is a  $GF(2)$ -vector space with addition  $S \boxplus T$  defined as follows.

1. If  $v \in V(G)$  and  $S \cap \{v_\phi, v_\chi, v_\psi\} = \emptyset$ , then  $(S \boxplus T) \cap \{v_\phi, v_\chi, v_\psi\} = T \cap \{v_\phi, v_\chi, v_\psi\}$ .
2. If  $v \in V(G)$  and  $T \cap \{v_\phi, v_\chi, v_\psi\} = \emptyset$ , then  $(S \boxplus T) \cap \{v_\phi, v_\chi, v_\psi\} = S \cap \{v_\phi, v_\chi, v_\psi\}$ .
3. If  $v \in V(G)$  and  $S \cap \{v_\phi, v_\chi, v_\psi\} = T \cap \{v_\phi, v_\chi, v_\psi\}$ , then  $(S \boxplus T) \cap \{v_\phi, v_\chi, v_\psi\} = \emptyset$ .
4. If  $v \in V(G)$  and  $S \cap \{v_\phi, v_\chi, v_\psi\}$  and  $T \cap \{v_\phi, v_\chi, v_\psi\}$  are nonempty and distinct, then the only element of  $(S \boxplus T) \cap \{v_\phi, v_\chi, v_\psi\}$  is the one element of  $\{v_\phi, v_\chi, v_\psi\}$  that is not included in either  $S$  or  $T$ .

Recall that the power set  $\mathcal{P}(W(G))$  is an algebra over  $GF(2)$ , with symmetric difference used for addition and intersection used for multiplication. Let  $\sigma : \mathcal{P}(W(G)) \rightarrow \mathcal{S}(W(G))$  be the  $GF(2)$ -linear map with  $\sigma(\{x\}) = \{x\} \forall x \in W(G)$ . Then for each  $v \in V(G)$ ,  $\{\emptyset, \sigma(\{v_\phi\}), \sigma(\{v_\chi\}), \sigma(\{v_\psi\})\}$  is a subspace of  $\mathcal{S}(W(G))$ ; and  $\mathcal{S}(W(G))$  is the direct product of these subspaces.

Recall also that the *cycle space*  $Z(M(IAS(G)))$  is the  $GF(2)$ -subspace of  $\mathcal{P}(W(G))$  consisting of the subsets of  $W(G)$  corresponding to sets of columns of  $IAS(G)$  that sum to 0.

**Definition 22** Let  $G$  be a looped simple graph. A *transverse cycle* of  $G$  is an element of  $\mathcal{L}(G) = \mathcal{S}(W(G)) \cap Z(M(IAS(G)))$ .

**Lemma 23** Suppose  $S \in \mathcal{S}(W(G))$ , and let  $S_\phi = \{v \in V(G) \mid v_\phi \in S\}$ ,  $S_\chi^\ell = \{\text{looped } v \in V(G) \mid v_\chi \in S\}$ ,  $S_\psi^\ell = \{\text{looped } v \in V(G) \mid v_\psi \in S\}$ ,  $S_\chi = \{\text{unlooped } v \in V(G) \mid v_\chi \in S\}$ , and  $S_\psi = \{\text{unlooped } v \in V(G) \mid v_\psi \in S\}$ . Then  $S \in \mathcal{L}(G)$  if and only if the following conditions are met:

1. For every  $v \in S_\chi \cup S_\psi^\ell$ ,  $|N(v) \cap (S - S_\phi)|$  is even.
2. For every  $v \in S_\phi \cup S_\chi^\ell \cup S_\psi$ ,  $|N(v) \cap (S - S_\phi)|$  is odd.
3. For every  $v \in V(G)$  with  $\{v_\phi, v_\chi, v_\psi\} \cap S = \emptyset$ ,  $|N(v) \cap (S - S_\phi)|$  is even.

**Proof.**  $S$  is a transverse cycle of  $G$  if and only if the sum of the columns of  $IAS(G)$  included in  $S$  is 0. That is,  $S \in \mathcal{L}(G)$  if and only if for every  $v \in V(G)$ ,  $S$  contains an even number of columns of  $IAS(G)$  with 1s in the  $v$  row. ■

**Proposition 24** Let  $\Phi(G) = \{v_\phi \mid v \in V(G)\}$ , and let  $\Psi(G) = \{v_\psi \mid v \in V(G) \text{ is looped}\} \cup \{v_\chi \mid v \in V(G) \text{ is unlooped}\}$ . Then

$$\mathcal{L}(G) = \{\sigma(X \cdot \Psi(G)) \boxplus \sigma(N(X) \cdot \Phi(G)) \mid X \subseteq V(G)\}.$$

**Proof.** This follows immediately from Lemma 23, with  $X = S - S_\phi$ . ■

$\Phi(G)$  and  $\Psi(G)$  are disjoint elements of  $\mathcal{S}(W(G))$ , and each is of size  $|V(G)|$ , so they satisfy Bouchet's definition of *supplementary vectors* [11]. It follows from Proposition 24 that  $\mathcal{L}(G)$  is an isotropic system with fundamental graph  $G$ . The basic theorem of isotropic systems – that two simple graphs are locally equivalent if and only if they are fundamental graphs of strongly isomorphic isotropic systems – now follows immediately from Corollary 18.

It is worth taking a moment to observe that even though  $\mathcal{L}(G)$  only includes the transverse cycles of  $G$ , it contains enough information to determine  $G$ , and hence also  $M(IAS(G))$ . The reason is simple: For each  $v \in V(G)$ ,  $\mathcal{L}(G)$  contains precisely one transverse cycle  $\zeta_v \subset \{v_\chi, v_\psi\} \cup \{w_\phi \mid v \neq w \in V(G)\}$ . The open neighborhood of  $v$  is  $N(v) = \{w \mid w_\phi \in \zeta_v\}$ , and  $v$  is looped if and only if  $v_\psi \in \zeta_v$ .

Before proceeding we observe that our results shed new light on Bouchet's work. Indeed, they contradict the first sentence of the following comment from [14]:

The theory of isotropic systems can be considered as an extension of the theory of binary matroids, whereas delta-matroids extend arbitrary matroids. However delta-matroids do not generalize isotropic systems.

In fact, the theory of isotropic systems can be considered as a *subset* of the theory of binary matroids, and so can the theory of graphic delta-matroids.

By the way, one does not need the results of this paper to contradict the second sentence of Bouchet's comment. If  $G$  is a looped simple graph then  $G$  is completely determined by  $D(G)$ : a vertex  $v$  is looped if and only if  $\{v\} \in D(G)$ , two looped vertices  $v$  and  $w$  are adjacent if and only if  $\{v, w\} \notin D(G)$ , and otherwise two vertices  $v$  and  $w$  are adjacent if and only if  $\{v, w\} \in D(G)$ . Consequently,  $D(G)$  also determines the isotropic systems with fundamental graph  $G$ , up to strong isomorphism. All isotropic systems have fundamental graphs, and there are non-graphic delta-matroids, so it would certainly seem that delta-matroids *do* generalize isotropic systems.

## 6 Interlace polynomials and Tutte polynomials

Motivated by problems that arise in the study of DNA sequencing, Arratia, Bollobás and Sorkin introduced a one-variable graph polynomial, the *vertex-nullity interlace polynomial*, in [2]. In subsequent work [3, 4] they observed that this one-variable polynomial may be obtained from the Tutte-Martin polynomial of isotropic systems studied by Bouchet [12, 15], introduced an extended two-variable version of the interlace polynomial, and observed that the interlace polynomials are given by formulas that involve the nullities of matrices over the two-element field,  $GF(2)$ . Inspired by these ideas, Aigner and van der Holst [1], Courcelle [19] and the author [30, 31] introduced several different variations on the interlace polynomial theme.

All these references share the underlying presumption that although the interlace and Tutte-Martin polynomials are connected to other graph polynomials in some ways, they are in a general sense separate invariants. In this section we point out that in fact, the interlace polynomials of graphs can be derived from parametrized Tutte polynomials of isotropic matroids.

One way to define the *Tutte polynomial* of  $M(IAS(G))$  is a polynomial in the variables  $s$  and  $z$ , given by the subset expansion

$$t(M(IAS(G))) = \sum_{T \subseteq W(G)} s^{r^G(W(G)) - r^G(T)} z^{|T| - r^G(T)}.$$

Here  $r^G$  denotes the rank function of  $M(IAS(G))$ . We do not give a general account of this famous invariant of graphs and matroids here; thorough introductions may be found in [5, 17, 20, 23].

Tutte polynomials of graphs and matroids are remarkable both for the amount of structural information they contain and for the range of applications in which they appear. Some applications (electrical circuits, knot theory, network reliability, and statistical mechanics, for instance) involve graphs or networks whose vertices or edges have special attributes of some kind – impedances and resistances in circuits, crossing types in knot diagrams, probabilities of failure and successful operation in reliability, bond strengths in statistical mechanics. A natural way to think of these attributes is to allow each element to carry two parameters,  $a$  and  $b$  say, with  $a$  contributing to the terms of the Tutte polynomial corresponding to subsets that include the given element, and  $b$  contributing to the terms of the Tutte polynomial corresponding to subsets that do not. Zaslavsky [38] calls the resulting polynomial

$$\sum_{T \subseteq W(G)} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) s^{r^G(W(G)) - r^G(T)} z^{|T| - r^G(T)} \quad (1)$$

the *parametrized rank polynomial* of  $M(IAS(G))$ ; we denote it  $\tau(M(IAS(G)))$ .

We do not give a general account of the theory of parametrized Tutte polynomials here; the interested reader is referred to the literature, for instance [6, 21, 28, 29, 38]. However it is worth taking a moment to observe that parametrized polynomials are very flexible, and the same information can be formulated in many ways. For instance if  $s$  and the parameter values  $b(w)$  are all invertible then formula (1) is equivalent to

$$s^{r^G(W(G))} \cdot \left( \prod_{w \in W(G)} b(w) \right) \cdot \sum_{T \subseteq W(G)} \left( \prod_{t \in T} \left( \frac{a(t)}{b(t)s} \right) \right) (sz)^{|T| - r^G(T)},$$

which expresses  $\tau(M(IAS(G)))$  as the product of a prefactor and a sum that is essentially a parametrized rank polynomial with only  $a$  parameters and one variable,  $sz$ . We prefer formula (1), though, because we do not want to assume invertibility of the  $b$  parameters.

Suppose that the various parameter values  $a(w)$  and  $b(w)$  are independent indeterminates, and let  $P$  denote the ring of polynomials with integer coefficients in the  $2+6|V(G)|$  independent indeterminates  $\{s, z\} \cup \{a(w), b(w) \mid w \in W(G)\}$ . Let  $J$  be the ideal of  $P$  generated by the set of  $4|V(G)|$  products  $\{a(v_\phi)a(v_\chi), a(v_\phi)a(v_\psi), a(v_\chi)a(v_\psi), b(v_\phi)b(v_\chi)b(v_\psi) \mid v \in V(G)\}$ , and let  $\pi : P \rightarrow P/J$  be the canonical map onto the quotient. Then the only summands of (1) that make nonzero contributions to  $\pi\tau(M(IAS(G)))$  correspond to subsets  $T \subseteq W(G)$  with the property that  $|T \cap \{v_\phi, v_\chi, v_\psi\}| = 1 \ \forall v \in V(G)$ . Each such  $T$  is a transversal of the canonical partition of  $W(G)$ , i.e.,  $T$  is a subset of  $W(G)$  with  $|T \cap \{v_\phi, v_\chi, v_\psi\}| = 1 \ \forall v \in V(G)$ . We denote the collection of all such transversals  $\mathcal{T}(W(G))$ . Each  $T \in \mathcal{T}(W(G))$  has  $|T| = |V(G)| = r^G(W(G))$ , so  $s$  and  $z$  have the same exponent in the corresponding term of (1):

$$\pi\tau(M(IAS(G))) = \pi \left( \sum_{T \in \mathcal{T}(W(G))} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^{|V(G)| - r^G(T)} \right).$$

Every generator of  $J$  involves the product of at least two different parameters  $a(v_i), b(v_i)$  corresponding to a single vertex  $v$  of  $G$ . It follows that  $\pi$  is injective when restricted to the additive subgroup  $A$  of  $P$  generated by products

$$\left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^k$$

where  $k \geq 0$  and  $T \in \mathcal{T}(W(G))$ . Consequently there is a well-defined isomorphism of abelian groups  $\pi^{-1} : \pi(A) \rightarrow A$ , and we have

$$\pi^{-1}\pi\tau(M(IAS(G))) = \sum_{T \in \mathcal{T}(W(G))} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^{|V(G)| - r^G(T)}. \quad (2)$$

Note that  $\pi^{-1}\pi\tau(M(IAS(G)))$ , the image of the parametrized Tutte polynomial  $\tau(M(IAS(G)))$  under the mappings  $\pi$  and  $\pi^{-1}$ , might also be described as the *section* of  $\tau(M(IAS(G)))$  corresponding to  $\mathcal{T}(W(G))$ . Either way, formula (2) describes an element of  $P$ , where  $s, z$  and the various parameter values  $a(w), b(w)$  are all independent indeterminates.

Arratia, Bollobás and Sorkin [4] define the two-variable *interlace polynomial*  $q(G)$  by the formula

$$\begin{aligned} q(G) &= \sum_{S \subseteq V(G)} (x-1)^{r(A(G)[S])} (y-1)^{|S| - r(A(G)[S])} \\ &= \sum_{S \subseteq V(G)} \left( \frac{y-1}{x-1} \right)^{|S| - r(A(G)[S])} (x-1)^{|S|}. \end{aligned}$$

Here  $r(A(G)[S])$  denotes the  $GF(2)$ -rank of the principal submatrix of  $A(G)$  involving rows and columns corresponding to vertices from  $S$ .



Let  $\mathcal{T}_0(W(G)) = \{T \in \mathcal{T}(W(G)) \mid v_\psi \notin T \ \forall v \in V(G)\}$ , and for  $T \in \mathcal{T}(W(G))$  let  $S(T) = \{v \in V(G) \mid v_\chi \in T\}$ . Then  $T \mapsto S(T)$  defines a bijection from  $\mathcal{T}_0(W(G))$  onto the power-set of  $V(G)$ . As  $r^G(T)$  is the  $GF(2)$ -rank of the matrix

$$(\text{columns } v_\phi \text{ with } v \notin S(T) \mid \text{columns } v_\chi \text{ with } v \in S(T))$$

and the columns  $v_\phi$  are columns of the identity matrix,

$$r^G(T) = |V(G)| - |S(T)| + r(A(G)[S(T)]).$$

It follows that  $q(G)$  may be obtained from  $\pi^{-1}\pi\tau(M(IAS(G)))$  by setting  $a(v_\phi) \equiv 1$ ,  $a(v_\chi) \equiv x - 1$ ,  $a(v_\psi) \equiv 0$ ,  $b(v_\phi) \equiv 1$ ,  $b(v_\chi) \equiv 1$ ,  $b(v_\psi) \equiv 1$ ,  $s = y - 1$  and  $z = 1/(x - 1)$ . These assignments are not unique; for instance the values of  $s$  and  $z$  may be replaced by  $s = (y - 1)/u$  and  $z = u/(x - 1)$  for any invertible  $u$ .

The reader familiar with the Tutte-Martin polynomials of isotropic systems studied by Bouchet [12, 15] and the interlace polynomials introduced by Aigner and van der Holst [1], Courcelle [19], and the author [31] will have no trouble showing that appropriate values for  $s$ ,  $z$  and the  $a$  and  $b$  parameters yield all of these polynomials from the parametrized rank polynomial  $\tau(M(IAS(G)))$ .

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